The basic household maximization problem

1 The model

One of the fundamental issues in economics with important implications for macroeconomic modeling is the optimal choice of consumption-savings by households. This decision taken by households is a fundamental piece in the construction of the micro-founded dynamic general equilibrium models that are used in current macroeconomic analysis. These models consider households or families as one of the economic agents in an economy. These agents make decisions on some control variables to maximize their objective function, which is what we call the utility function. This utility function can depend on a wide set of variables, the main one of which is consumption. Here, we solve a basic household maximization problem where utility only depends on consumption. The utility function is strictly increasing, strictly concave, and twice differentiable.

In the neoclassical framework it is assumed that economic agents are rational and that they maximize their utility function throughout their life cycle. We also assume that the capital markets are perfect, that is, the savings of the agent can be positive (credit position) or negative (debit position), in each moment of time, with no-liquidity constraints. From this decision problem, we simultaneously determine two of the most relevant macroeconomic variables to explain the behavior of an economy: the level of consumption, which is the main component of aggregate demand in quantitative terms, and the level of savings, which will determine the level of investment and, therefore, the process of capital accumulation and future output. We also assume that the utility is instantaneous and only depends on the level of consumption of the period, so it is additively separable over time. In this way, we obtain the intertemporal consumption-savings decision, since through the saving decision the agent can separate its consumption profile from its income profile period by period, determining its levels of future consumption based on a current saving decision. The objective of households is to maximize the sum of discounted utility for each period of their life cycle, where the weight to future utility is lower compared to current utility. We assume a finite-life household with a lifespan of T periods.

The basic household maximization problem consist in solving the following problem:

$$\max_{\{C_t\}_{t=0}^{T}} E_t \sum_{t=0}^{T} \beta^t U(C_t) \tag{1}$$

subject to the budget constraint given by:

$$C_t + B_t = W_t L_t + (1 + R_{t-1}) B_{t-1} \tag{2}$$

where C_t is consumption, B_t is the stock of savings or amount of financial assets (which can be positive or negative) in the current period, R_{t-1} is the interest rate applied to financial assets (savings) in the previous period, L_t is labor, and W_t is labor income. We assume that labor, wage and the interest rate are exogenous. E_t is the mathematical expectation operator. We assume perfect foresight, which means that we have information about the future value of all the variables. This means that we can directly eliminate the mathematical expectation of the problem to be maximized. Labor is normalized to one.

Two additional restrictions, initial and final stock of financial assets, are required. Initial condition is assumed to be $B_{-1} = 0$, and therefore, the amount of financial assets in period $t = 0$ is equivalent to the savings in that period, $B_0 = W_0 - C_0$. Second, the amount of financial assets in the final period B_T is also zero, given the assumption of finite life. The discount factor is defined by $0 < \beta < 1$. This discount rate depends on the intertemporal subjective preference rate, denoted by $\theta > 0$, such as $\beta = 1/(1 + \theta)$.

Let's assume that the utility function has a logarithmic form. In this case, the problem of the consumer would be given by:

$$\max_{\{C_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \ln C_t \tag{3}$$

subject to budget constraint and initial and final conditions. The auxiliary Lagrange function is:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \ln C_t - \lambda_t \left[C_t + B_t - W_t - (1 + R_{t-1}) B_{t-1}\right] \tag{4}$$

First-order conditions are the following:

$$\frac{\partial \mathcal{L}}{\partial C_t} : \beta^t \frac{1}{C_t} - \lambda_t = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial B_t} : -\lambda_t + \lambda_{t+1}(1 + R_t) = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} : C_t + B_t - W_t - (1 + R_{t-1})B_{t-1} = 0 \tag{7}$$

By operating we obtain the following optimal path for consumption:

$$C_{t+1} = \beta(1 + R_t)C_t \tag{8}$$

Steady State Steady state variables are denoted by an upper bar. The steady state is given by that value for the level of consumption that remains constant over time. So, if $C_t = C_{t+1} = ... = \overline{C}$, then it turns out that $1 = \beta(1 + \overline{R})$, and implying that the steady-state value for the interest rate is:

$$\overline{R} = \frac{1 - \beta}{\beta} = \theta \tag{9}$$

that is, in a steady state, the interest rate would be equal to the intertemporal subjective preference rate.

This optimization problem can be solved using the "Solver" tool of Excel by first calibrating the parameters and the exogenous variables, and introducing a guess solution for consumption. Hence, no analytical solution is needed and the initial equations of the model (utility function, the budget constraint and initial and final conditions) can directly be introduced in Excel.

2 Taking the model to Excel

This model is solved in the Excel file named "Household-1.xlsx". For the calibration of the model we only need one parameter: the intertemporal discount rate, β. We also need to determine two exogenous variables: the real interest rate, R, and the income level, W. Both the intertemporal discount rate and the interest rate are determining factors of the optimal consumption path, together with the particular functional form of the utility function, as we have shown analytically. In practice, the real interest rate is an endogenous variable reflecting the marginal productivity of physical capital. However, for now, let's assume that this exogenous variable remains constant. On the other hand, we must also specify the salary income of the individual in each period of their life, which we have assumed to be exogenous but in practice is also an endogenous variable representing marginal productivity of labor. In this first exercise, we will assume that the income is constant for all periods. The values we set are an intertemporal discount factor of 0.97, which appears in cell "B4", while in cell "B7" the real interest rate of 2% ($R = 0.02$) appears, and a value for the salary income of $W = 10$, which it is included as a column.

The variables that we need to define to solve this problem are the following. The "D" column is the time index, while the "E" column will show the optimal consumption values for each period, which are the variables that we have to calculate. That is, the solution to our problem would appear in this column. Let's suppose that the household lives from period 0 to period 30 ($T = 30$). Column "F" is the exogenous (labor) income, which we assume is given and constant period to period, column "G" is the savings obtained as the difference between consumption and income of each period, and finally, column "H" shows the value for discounted utility.

To solve the exercise using Excel's "Solver" tool, we operate as follows. First, we fill with fictitious values (a guess solution) the column corresponding to consumption (column "E"). This is what is known as the "seed", and will be the initial values that the algorithm in Excel will use to obtain the solution to the problem. These initial values should be as close as possible to the final solution. The closer they are to the final solution, the easier it will be for the computer to calculate the correct solution. On the other hand, if the initial values we propose are very different from the final solution, we could find the case in which the algorithm that solves our system of equations is not able to find the correct solution. It is important to keep in mind that these fictitious values (guess solution) that we initially provide cannot be very different from the final solution. From here, and in an iterative process, the algorithm changes these initial values obtaining new solutions until reaching the final value. A hint for the "seed" is to introduce a constant value for consumption for each period such as the stock of saving at the final period be close to zero.

Next, column "G" shows the savings, in terms of the stock of financial assets. The savings of the first period is simply the difference between the level of consumption and the level of income in that period. Thus, if we place the cursor in cell "G3", this expression appears:

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=F3-E3
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that is, wage income (column "F") minus consumption (column "E") in period 0. On the contrary, if we place the cursor in cell "G4" we see that the expression that appears is:

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=(1+Rbar)*G3+F4-E4
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that is, it is the income generated by the savings made up to the previous period plus the income of the period minus consumption of the period. The following rows in this column contain this same expression. That is, it is the gross yields from savings made up to the previous period plus the new savings of the period.

Finally, column "H" presents the value of the discounted utility in each period. If we place the cursor in cell "H3", the expression appears:

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=Beta^D3*LN(E3)
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which is the valuation of the utility in period 0, that is, the logarithm of the consumption multiplied by the intertemporal discount factor to the time index power corresponding to that period. Finally, in cell "H34" appears the sum of the discounted utilities, which is the value that we have to maximize. Once we have this information, then we go to the "Solver" tool and enter the data corresponding to the problem we want to solve. In our case, the target cell is "H34", changing the cells of the variables from "E3" to "E33" and subject to the restriction that cell "G33" must be greater than or equal to zero. Once these steps have been completed, we can execute the "Solver" tool, pressing the "Solve" button, and we will obtain the solution for the optimal consumption path in the "E" column.

To execute Excel's "Solver" to solve our maximization problem, we have to go in the main menu to "Tools" or "Data", depending on the version of Excel that we are using, and select "Solver" and a new window will be open. The "Solver" tool may not come incorporated directly into the Excel menu, so we probably have to install it. If the solver tool is not already installed, then we have to go to "File", "Options" and "Add-ons", to install the tool, or go to the Office Button and select "Excel Options", "Add-ons" and select "Solver" for installation, depending on the version we are using.

When selecting "Solver", a dialogue window appears in the spreadsheet with the title "Solver Parameters". The first item that appears in the "Solver" dialog window is "Objective cell:" or "Set Objective:". This option refers to the value of the objective function of the problem that we want to solve. In our case, it will refer to the total utility of the individual throughout his life. Specifically, it would be the discounted sum of the utility obtained by the individual in each period of his life cycle.

Then, the instruction "Objective cell value:" appears in which there are three options: "Maximum", "Minimum" and "Value of:". These options refer to the type of problem we want to solve. If we want to maximize the value of the target cell, we would select "Maximum". If what we want is to minimize a certain problem, then we would select the "Minimum" option. On the contrary, if what we want is that it reaches a certain value, we would introduce this value in the option "Value of:". In our case, the problem we want to solve consists in maximizing the utility of the consumer, so we would select the option "Maximum", which is precisely the option marked by default.

Then, the instruction "By changing variable cells" appears. Here, we have to introduce the cells in which Excel will calculate the objective variable, that is, the level of consumption period by period. The spreadsheet will present the solution to the problem that we want to solve in the cells that we indicate in this section. In our specific case, we will obtain the level of consumption in each period that maximizes the total utility of the individual throughout his life. For the construction of the problem, we have previously proposed a tentative value for these cells.

Finally, the instruction "Subject to the Constraints:" appears. In this section, we must introduce the constraints to which the problem we want to solve is subject. The constraint with which our problem is going to count is that the amount of financial assets (savings) of the individual at the end of his life must be zero ($B_T = 0$).

To the right of the "Solver" dialog box appears a tab called "Options". If we click on this tab, a new dialogue box appears. Again, the exact format of this dialog depends on the version of Excel that we are using. This table allows to change different parameters, such as the maximum calculation time, the number of iterations, the precision, the tolerance and the convergence. It also includes other additional options. Finally, there are some options related to the estimation, calculation of derivatives and regarding the search algorithm of the solution. However, the dialog boxes of the "Solver" can be very different depending on the version of Excel that we are using. Once we have it all, we simply give the tab to "Solve" and automatically we get the solution to the problem (if Excel has not found any problem to solve it). Excel will determine the values of the level of consumption for each period such that the sum of the discounted profits is the maximum possible.

3 Exercises

1. Using the "Household-1.xlsx" spreadsheet, study the effects of an increase in the real interest rate. In particular, suppose that the interest rate increases to 5 percent, $R = 0.05$. How is the slope of the optimal path of consumption now? What happens now with the savings? What are these changes in the optimal decision of the consumer?

2. Suppose now that the interest rate is 0 percent, $R = 0.00$. What consequences does it have on the consumption-savings decision?

3. Given the parameters calibrated in the spreadsheet "Household-1.xlsx", what should be the interest rate so that the level of consumption is the same for all periods? How does this value of the interest rate relate to the intertemporal preference rate?

4. Using the spreadsheet "Household-1.xlsx", suppose that the exogenous income is increasing over time. For example assume that initial wage income is 10 and it increases in 1 unit each period. Obtain the new optimal consumption path.