Household maximization problem with labor decision

1 The model

Typically, in micro-founded dynamic general equilibrium macroeconomic models, the household's utility function depends positively on two arguments: consumption and leisure, leisure being understood as non-working time. In this context, households obtain satisfaction from both consumption and leisure. However, in order to consume goods and services, you must first generate income and, therefore, you must renounce part of your leisure time and dedicate that time to production activities (work). From this utility function, we obtain two fundamental decisions that will determine the behavior of an economy. The first is an intertemporal decision between consumption-savings, and the second is the consumption-leisure intratemporal decision, from which we will determine the labor supply.

The fundamental idea of this utility function is that agents have an endowment of available time that they can devote to different activities. The part of the available time that they wish to assign to working activities will give rise to the labor supply, understanding this as a renunciation of leisure. This is a static decision, given that the available time cannot be accumulated. That is, the decision to work or not (what fraction of the available time we will devote to work) is taken period by period. The time dedicated to work generates a disutility. We assume a finite-life household with a lifespan of T periods.

The basic household maximization problem consist in solving the following problem:

$$\max_{\{C_t,O_t\}_{t=0}^{T}} E_t \sum_{t=0}^{T} \beta^t U(C_t, O_t) \tag{1}$$

subject to the budget constraint given by:

$$C_t + B_t = W_t L_t + (1 + R_{t-1}) B_{t-1} \tag{2}$$

where C_t is consumption, O_t is leisure, B_t is the stock of savings or amount of financial assets, R_{t-1} is the interest rate applied to financial assets (savings) in the previous period, L_t is labor, and W_t is labor income. We assume that the interest rate is exogenous. E_t is the mathematical expectation operator. We assume perfect foresight, and hence, we can directly eliminate the mathematical expectation of the problem to be maximized. Total available time is normalized to one. Hence, labor can be defined as,

$$L_t = 1 - O_t \tag{3}$$

Two additional restrictions, initial and final stock of financial assets, are required. Initial condition is assumed to be $B_{-1} = 0$, and therefore, the amount of financial assets in period $t = 0$ is equivalent to the savings in that period, $B_0 = W_0 - C_0$. Second, the amount of financial assets in the final period B_T is also zero, given the assumption of finite life. The discount factor is defined by $0 < \beta < 1$. This discount rate depends on the intertemporal subjective preference rate, denoted by $\theta > 0$, such as $\beta = 1/(1 + \theta)$.

Let's assume the following utility function,

$$U(C_t, 1 - L_t) = \gamma \ln C_t + (1 - \gamma) \ln(1 - L_t) \tag{4}$$

where $\gamma \in (0, 1)$, is a preferences parameter representing the weight of consumption in the household's utility. In this case, the problem of the consumer would be given by:

$$\max_{\{C_t, L_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \left[\gamma \ln C_t + (1 - \gamma) \ln(1 - L_t)\right] \tag{5}$$

subject to budget constraint and initial and final conditions. The auxiliary Lagrange function is:

$$\mathcal{L} = \sum_{t=0}^{T} \left[\begin{array}{c} \beta^t \left[\gamma \ln C_t + (1 - \gamma) \ln(1 - L_t)\right] \\ -\lambda_t \left(C_t + B_t - W_t L_t - (1 + R_{t-1}) B_{t-1}\right) \end{array} \right] \tag{6}$$

First-order conditions are the following:

$$\frac{\partial \mathcal{L}}{\partial C_t} \quad : \quad \beta^t \gamma \frac{1}{C_t} - \lambda_t = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial L_t} \quad : \quad -\beta^t (1-\gamma) \frac{1}{1-L_t} + \lambda_t W_t = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial B_t} \quad : \quad \lambda_{t+1}(1+R_t) - \lambda_t = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} \quad : \quad C_t + B_t - W_t L_t - (1+R_{t-1})B_{t-1} = 0 \tag{10}$$

By operating we obtain the following optimal path for consumption:

$$C_{t+1} = \beta(1+R_t)C_t \tag{11}$$

and the following optimal labor supply:

$$\frac{1-\gamma}{\gamma} \frac{C_t}{1-L_t} = W_t \tag{12}$$

Steady State Steady state variables are denoted by an upper bar. The steady state is given by that value for the level of consumption that remains constant over time. So, if $C_t = C_{t+1} = ... = \overline{C}$, then it turns out that $1 = \beta(1+\overline{R})$, and implying that the steady-state value for the interest rate is:

$$\overline{R} = \frac{1-\beta}{\beta} = \theta \tag{13}$$

that is, in steady state, the interest rate would be equal to the intertemporal subjective preference rate. Steady state labor is given by:

$$\overline{L} = \gamma \tag{14}$$

This optimization problem can be solved using the "Solver" tool of Excel by first calibrating the parameters and the exogenous variables, and introducing a guess solution for consumption and labor. Hence, no analytical solution is needed and the initial equations of the model (utility function, the budget constraint and initial and final conditions) can directly be introduced in Excel.

2 Taking the model to Excel

To numerically compute the previous problem, we will use the Excel "Solver" tool, similar to what we did in the previous chapter. In this case, we use this tool to calculate the optimal values of both the consumption and the proportion of time that the agent will dedicate to working activities. The file where we have solved this problem is called "Household-2.xls".

As we can see, in this case, we need to calibrate two parameters: the intertemporal discount rate, β, and the weight to consumption in the utility function, γ. In addition, we have two exogenous variables: the real interest rate and the salary per unit of time. The values we have set are an intertemporal discount factor of 0.97, a value that appears in cell "B4" and a weight for consumption in the utility function of 40%, a value that appears in cell "B5". Additionally, we will assume that the real interest rate is 2%, which corresponds to cell "B8", and a salary per unit of time of 30.

If the interest rate is greater than $1/\beta - 1$ (from the steady state condition), then the optimal path of consumption is increasing over time, being decreasing in the opposite case. In our example, the interest rate is 0.02, whereas $1/\beta - 1 = 0.031$ so the resulting optimal path will be decreasing in time.

The rest of the information we need appears in columns "D" through "I". Column "D" represents the time, while column "E" will present the consumption values, which together with column "F", that is the labor supply, are the variables that we have to calculate to solve the optimization problem. "G" is labor income, which is obtained by multiplying the salary per unit of time, which is assumed exogenous, by the working time optimal decision, column "H" is the savings obtained as the difference between consumption and income of each period and finally column "I" shows the satisfaction of the individual based on consumption in updated terms. The values that we have to introduce in columns "E" and "F" to solve this problem, are values that we assume are close to the final solution, and constitute the initial values of the maximization algorithm that the "Solver" tool will apply.

Savings are calculated in the following way. Saving in the first period is simply calculated as the difference between the level of consumption and the level of income in that period. Thus, if we place the cursor in cell "H3", we see that the expression appears:

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=G3-E3
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that is, the wage income (column "G") minus consumption (column "E") in period 0.

On the contrary, if we place the cursor in cell "H4", we see that the expression that appears is:

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=(1+R_0)*H3+G4-E4
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that is, the gross returns to savings made up to the previous period plus the income of the period less the consumption of the period. The following rows in this column contain this same expression. That is, the gross returns of the savings made up to the previous period plus the new savings of the period.

Finally, column "I" presents the valuation of the discounted utility in each period. If we place the cursor in cell "I3", the expression appears:

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=Beta^D3*(Gamma*LN(E3)+(1-Gamma)*LN(F3))
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which is the valuation of the utility in period 0, that is, the logarithm of the consumption multiplied by the intertemporal discount factor raised to the time index corresponding to that period. Finally, in cell "I34" appears the sum of the discounted utilities, which is the value that we have to maximize.

To solve the exercise using the Excel Solver tool, please see the instructions in the previous note.we operate as follows. First of all, we fill with fictitious values (a guess solution) the column corresponding to consumption and the one corresponding to labor. For example, we can give a value of 0.35 to labor (this supposes that 35% of the available time is dedicated to working activities), and as an initial guess we can suppose that the consumption is equal to the income. To execute this Excel tool, we have to go to "Tools" and select "Solver". The first element that appears in the window is "Objective cell:". This option refers to the value of the objective function of the problem that we want to solve. In our case, it refers to the total utility of the individual throughout their life. Specifically, it would be the discounted sum of the utility obtained by the individual in each period of their life, and would be given in cell "I34".

Then, the instruction "By Changing Variable Cells" appears. Here we have to introduce the cells in which Excel will calculate the objective variables, that is, the level of consumption and the level of working time, period by period. In the cells that we indicate in this section is where the spreadsheet will present the solution to the problem that we want to solve. In our specific case, we will obtain the level of consumption in each period that maximizes the total utility of the household throughout their life, as well as the job offer (the proportion of time dedicated to work) in each period. The expression we would have to enter is "`$E3:F33$`". Finally, the instruction "Subject to the constraints:" appears. In this section, we must introduce the restrictions to which the problem we want to solve is subject. This constraint is the quantity of assets (savings) of the agent at the end of their life must be zero.

3 Exercises

1. Using the spreadsheet "Household-2".xlsx, study the effects of an increase in the real interest rate. In particular, suppose that the interest rate increases to 5 percent, $R = 0.05$. How is the slope of the optimal path of consumption now? What happens with labor supply?

2. Analyze the effects of a change in the parameter γ (for instance, consider that γ changes from 0.4 to 0.5).

3. Suppose that the interest rate is 0 percent, $R = 0.00$. What consequences does it have on the consumption-savings decision and the consumption-leisure decision?

4. Suppose that there is a maximum limit to work for which the fraction of work time cannot be greater than 40% of the available time. Evaluate what the effects of this limit are (Hint: for this, you have to introduce an additional restriction in the problem to be solved such that the column corresponding to the employment cannot be higher than 0.4: "`$F$3:$F$33<=0.4`").