The Basic Real Business Cycle model

1 The model

Current macroeconomic analysis is carried out mainly through the construction, calibration, estimation and simulation of dynamic general equilibrium models, both stochastic and deterministic. These models represent an unified theoretical framework that can be used for the study of the economy both in the short term (business cycles) and in the long term (economic growth). The key characteristic of this typology of models is that they are micro-founded. For the construction of this micro-founded macroeconomic model, we set up a macroeconomic environment where both households and firms takes economic decisions. This basic framework can be extended by considering other economic agents, such as the government, the central bank, the financial sector, etc. Here, we will solve a basic dynamic general equilibrium model. This is a deterministic version of the Real Business Cycle (RBC) model that has been extensively used in macroeconomic analysis. The model represents an economy composed of two types of economic agents: households and firms. The model includes as an additional variable, the total factor productivity (TFP), which represents aggregate productivity of the economy or neutral technological progress in the sense of Hicks.

Households The household's maximization problem is given by:

$$\max_{\{C_t\}_{t=0}^{T}} E_t \sum_{t=0}^{T} \beta^t \ln C_t \tag{1}$$

where C_t is consumption, $0 < \beta < 1$ is the intertemporal discount factor, and $E_t(\cdot)$ is the mathematical expectation on future variables. Since we consider a context without uncertainty, that is, with perfect foresight, we can eliminate the expectation operator from the maximization problem, given that the value of all the variables in the future is known at the present time.

Households maximize the weighted sum of their profits subject to the budget constraint

$$C_t + I_t = W_t L_t + R_t K_t \tag{2}$$

where I_t is investment, W_t is wage, L_t is labor, R_t is returns to capital, and K_t is the capital stock. Investment is accumulated in the form of physical capital from the following process:

$$K_{t+1} = (1-\delta) K_t + I_t \tag{3}$$

where $\delta > 0$ is the rate of physical depreciation of capital.

Finally, we need to determine the stock of initial capital, K_0, as well as the stock of final capital, K_{T+1}, in the case in which the life cycle of the household is finite. We consider either a finite life cycle (for the case in which we use the Solver tool in Excel), or an infinite life cycle, for the case in which we compute numerically a linear approximation to the model.

The household maximization problem can by defined by the following Lagrange auxiliary function,

$$\mathcal{L} = \sum_{t=0}^{T} \left[\beta^t \ln C_t - \lambda_t (C_t + K_{t+1} - W_t L_t - (R_t + 1 - \delta) K_t) \right] \tag{4}$$

where households take as given the relative prices of the productive factors. First-order conditions, for $t = 0, 1, 2, \ldots, T$, are given by:

$$\frac{\partial \mathcal{L}}{\partial C_t} \; : \; \beta^t \frac{1}{C_t} - \lambda_t = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} \; : \; \lambda_{t+1} \left[R_{t+1} + 1 - \delta \right] - \lambda_t = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} \; : \; C_t + K_{t+1} - (R_t + 1 - \delta) K_t - W_t L_t = 0 \tag{7}$$

Solving for the Lagrange's multiplier (the shadow price of consumption), the following optimal consumption path is obtained,

$$C_{t+1} = \beta \left[R_{t+1} + 1 - \delta \right] C_t \tag{8}$$

The firms The other economic agent that we consider are firms, which represents the productive sector of the economy. We assume that firms maximize profits subject to the technological restriction represented by a constant return to scale technology. We assume a competitive environment. These assumptions results in zero profits, and hence, the factors will be remunerated based on their contribution to the production process.

We assume that the function of aggregate production (technology) has a Cobb-Douglas form:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{9}$$

where Y_t is the level of aggregate production of the economy, A_t is the total factor productivity (TFP) and α is the elasticity of output with respect to capital. We assume the TFP is determined exogenously from the following process:

$$A_t = A_{t-1}^{\rho} \varepsilon_t \tag{10}$$

where $\rho < 1$ is an autoregressive parameter that measures the persistence of shocks that affect the TFP, and ε_t is a disturbance term, which we can considered either stochastic and deterministic. In our case, we will consider it as an exogenous deterministic variable, whose value is one, except at the moment in which a technological shock occurs, taking a value different from one (higher than one for a shock that increases aggregate productivity and lower than one for a shock decreasing aggregate productivity).

The problem solved by the firm consists in maximizing profits, such that:

$$\max \Pi_t = A_t K_t^{\alpha} L_t^{1-\alpha} - W_t L_t - R_t K_t \tag{11}$$

The first-order conditions of the previous problem are:

$$\frac{\partial \Pi_t}{\partial K_t} : \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - R_t = 0 \tag{12}$$

$$\frac{\partial \Pi_t}{\partial L_t} : (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} - W_t = 0 \tag{13}$$

resulting:

$$R_t = \frac{\alpha A_t K_t^{\alpha} L_t^{1-\alpha}}{K_t} = \alpha \frac{Y_t}{K_t} \tag{14}$$

$$W_t = \frac{(1-\alpha) A_t K_t^{\alpha} L_t^{1-\alpha}}{L_t} = (1-\alpha) \frac{Y_t}{L_t} \tag{15}$$

Equilibrium of the model Competitive equilibrium consists of finding sequences of the $\{C_t, I_t, K_t, R_t, W_t, Y_t, A_t\}_{t=0}^{T}$ variables such that the conditions that define equilibrium (households maximize utility, firms maximize profits, and the feasibility condition of the economy holds) are satisfied. In summary, the model of our economy would be composed of the following seven equations:

$$C_{t+1} = \beta \left[R_{t+1} + 1 - \delta\right] C_t \tag{16}$$

$$R_t = \alpha \frac{Y_t}{K_t} = \frac{\alpha A_t K_t^{\alpha}}{K_t} = \alpha A_t K_t^{\alpha-1} \tag{17}$$

$$W_t = (1-\alpha) Y_t = (1-\alpha) A_t K_t^{\alpha} \tag{18}$$

$$Y_t = A_t K_t^{\alpha} \tag{19}$$

$$K_{t+1} = (1-\delta) K_t + I_t \tag{20}$$

$$C_t + I_t = Y_t \tag{21}$$

$$A_t = A_{t-1}^{\rho} \varepsilon_t \tag{22}$$

Solution: The dynamic system The previous system can be reduced to a system of two dynamic equations, one for consumption and another for the stock of capital, plus the equation that determines the behavior of the TFP,

$$C_{t+1} = \beta \left[\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta\right] C_t \tag{23}$$

$$K_{t+1} = (1-\delta) K_t + A_t K_t^{\alpha} - C_t \tag{24}$$

Steady state Steady state of the economy is defined by:

$$\overline{R} = \frac{1 - \beta + \beta\delta}{\beta} \tag{25}$$

$$\overline{K} = \left(\frac{(1 - \beta + \beta\delta)}{\alpha\beta\overline{A}}\right)^{\frac{1}{\alpha-1}} \tag{26}$$

$$\overline{Y} = \overline{A}^{\frac{1}{1-\alpha}} \left[\frac{\alpha\beta}{(1 - \beta + \beta\delta)}\right]^{\frac{\alpha}{1-\alpha}} \tag{27}$$

$$\overline{I} = \delta \left(\frac{(1 - \beta + \beta\delta)}{\alpha\beta\overline{A}}\right)^{\frac{1}{\alpha-1}} \tag{28}$$

$$\overline{C} = \overline{A}^{\frac{1}{1-\alpha}} \left[\frac{\alpha\beta}{(1-\beta+\beta\delta)} \right]^{\frac{\alpha}{1-\alpha}} - \delta \left(\frac{(1-\beta+\beta\delta)}{\alpha\beta\overline{A}} \right)^{\frac{1}{\alpha-1}} \tag{29}$$

$$\overline{A} = 1 \tag{30}$$

Log-linearized model The log-linear approximation to the model is given by,

$$\widehat{y}_t = \alpha \widehat{k}_t \tag{31}$$

$$\frac{1-\beta+\beta\delta-\alpha\beta\delta}{\alpha\beta}\widehat{c}_t + \widehat{k}_{t+1} = (1-\delta)\widehat{k}_t + \frac{(1-\beta+\beta\delta)}{\beta}\widehat{k}_t \tag{32}$$

$$\widehat{c}_{t+1} - \widehat{c}_t = (1-\beta+\beta\delta)(\alpha-1)\widehat{k}_{t+1} \tag{33}$$

$$\widehat{i}_t = \frac{1-\beta+\beta\delta}{\alpha\beta\delta}\widehat{y}_t - \frac{1-\beta+\beta\delta-\alpha\beta\delta}{\alpha\beta\delta}\widehat{c}_t \tag{34}$$

$$\widehat{a}_t = \rho \widehat{a}_{t-1} \tag{35}$$

where $\widehat{x}_t = \ln X_t - \ln \overline{X}$.

Solution: Log-linear dynamic system Operating, the log-linearized model can be represented by the following system of linear difference equations:

$$\begin{bmatrix} \Delta\widehat{c}_t \\ \Delta\widehat{k}_t \end{bmatrix} = \begin{bmatrix} -\frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)} & \frac{(\alpha-1)\Omega}{\beta(1+n)} \\ -\frac{\Gamma}{\alpha\beta(1+n)} & \frac{1-\beta(1+n)}{\beta(1+n)} \end{bmatrix} \begin{bmatrix} \widehat{c}_t \\ \widehat{k}_t \end{bmatrix} \tag{36}$$

where

$$\Omega = 1-\beta+\beta\delta \tag{37}$$

$$\Phi = 1-\beta+(1-\alpha)\beta\delta \tag{38}$$

Eigenvalues Once we have the model in log-linear terms, we can proceed to analyze its stability and obtain the eigenvalues associated with it. As we can see, we have transformed a system of non-linear dynamic equations into a linear dynamic system, in terms of the deviations (in logarithmic terms, that is, in percentage) of each variable with respect to the steady state. For this, we calculate:

$$Det\begin{bmatrix} -\frac{(\alpha-1)\Omega\Phi}{\alpha\beta}-\lambda & \frac{(\alpha-1)\Omega}{\beta} \\ -\frac{\Phi}{\alpha\beta} & \frac{1-\beta}{\beta}-\lambda \end{bmatrix}=0 \tag{39}$$

From the previous system, we obtain the following second order equation:

$$\lambda^2+\left(\frac{(\alpha-1)\Omega\Phi}{\alpha\beta}-\frac{1-\beta}{\beta}\right)\lambda-\left(\frac{1-\beta}{\beta}\right)\left(\frac{(\alpha-1)\Omega\Phi}{\alpha\beta}\right)+\left(\frac{(\alpha-1)\Omega}{\beta}\right)\left(\frac{\Phi}{\alpha\beta}\right)=0 \tag{40}$$

or equivalently:

$$\lambda^2+\frac{(\alpha-1)\Omega\Phi-\alpha(1-\beta)}{\alpha\beta}\lambda+\frac{(\alpha-1)\Omega\Phi}{\alpha\beta}=0 \tag{41}$$

The solution of the system is a saddle point. Eigenvalues are,

$$\lambda_1,\lambda_2=\frac{-\frac{(\alpha-1)\Omega\Phi-\alpha(1-\beta)}{\alpha\beta}\pm\sqrt{\left(\frac{(\alpha-1)\Omega\Phi-\alpha(1-\beta)}{\alpha\beta}\right)^2-4\frac{(\alpha-1)\Omega\Phi}{\alpha\beta}}}{2} \tag{42}$$

Jump forward-looking variable To compute the model, we need to calculate the short-term effect, that is, the variation in consumption (which is the "jumping" variable) just at the moment when a disturbance occurs. As we have seen in theoretical terms, when a disturbance occurs, the consumption is adjusted immediately until reaching the stable saddle path.

The dynamic equation obtained previously for consumption is:

$$\Delta\widehat{c}_t=-\frac{(\alpha-1)\Omega\Phi}{\alpha\beta}\widehat{c}_t+\frac{(\alpha-1)\Omega}{\beta}\widehat{k}_t \tag{43}$$

On the other hand, the stable path is defined by the trajectory:

$$\Delta\widehat{c}_t=\lambda_1\widehat{c}_t \tag{44}$$

Matching both expressions results in:

$$-\frac{(\alpha-1)\Omega\Phi}{\alpha\beta}\widehat{c}_t+\frac{(\alpha-1)\Omega}{\beta}\widehat{k}_t=\lambda_1\widehat{c}_t \tag{45}$$

The value of the jumping variable (the forward-looking variable, i.e., consumption) in the instant when a shock hits the economy is given by,

$$\widehat{c}_t=\frac{\alpha(\alpha-1)\Omega}{(\alpha-1)\Omega\Phi+\alpha\beta\lambda_1}\widehat{k}_t \tag{46}$$

where $\lambda_1$is the stable eigenvalue, in order the economy be at the stable saddle path to the new steady state.

2 Taking the model to Excel

The model can be solved using two alternative methods. Once the model has been solved analytically and the corresponding dynamic system has been obtained, then we will solve it computationally in Excel using the "Solver" tool. As can be observed in the Excel file "RBC-1.xlsx", we need to define first the value of the parameters of the model, which appear in cells "B4" to "B7". From these parameters and the steady-state expressions calculated above, we can obtain the steady-state values for the model variables, which appear in cells "B10" to "B15". If we place the cursor in cell "B10", the expression that appears is:

```
=PTF*((1-Beta+Delta*Beta)/(Alpha*PTF*Beta))^(Alpha/(Alpha-1))
```

which is the one corresponding to the value of the steady-state production. Similarly, in cell "B12" we have introduced the expression corresponding to the steady-state value of the stock of capital, so the expression that appears in said cell is:

```
=((1-Beta+Delta*Beta)/(Alpha*PTF*Beta))^(1/(Alpha-1))
```

Similarly, in cell "B13" we calculate the steady-state value of the consumption, in cell "B14" the steady-state value of the investment and in cell "B15" the steady-state value of the interest rate. Finally, in cell "B18", we assign the value of the technological change that we assume occurs in period 1, taking a zero value initially.

The variables of the model are defined in the columns "D-K", where the values corresponding to the initial steady state appear in the period 0. The column "D" is the time index, the column "E" is the TFP, while the column "F" gives us the optimal path of consumption, which is the variable that we have to calculate. Column "G" is the investment, which is simply the difference between what is produced and what is consumed, column "H" is production, column "I" is the stock of capital, column "J" is the return on capital and finally the "K" column is the discounted utility. In cell "E4", the expression

```
=E3^Rho*Epsilon
```

appears with the objective of simulating a productivity shock in period 1. In cell "E5" the introduced expression is "=E4^Rho", since we assume that the shock takes a positive or negative value at time 1 and zero in the following periods. This expression is copied to the following cells in the column.

In cell "I3", the initial capital stock appears. For its part, in cell "I4" appears the expression:

```
=(1-Delta)*I3+G3
```

where the stock of capital in each period of time is the stock of capital of the previous period, discounting the depreciation, plus the new capital that is incorporated, which is determined by savings. Finally, column "K" presents the value of the utility in discounted terms.

The sum of the discounted utilities is calculated in cell "K34", which will be the target cell to be maximized in the "Solver" tool. The solution to the problem is obtained by executing the "Solver", once we have defined the target cell to be maximized (the "K34"), the final condition (`"$I$34=K0"`), and the cells to change with the solution (`"$F$4:$F$33"`).

As we can now see, the optimal path of consumption that we obtain is completely horizontal, indicating that the consumption is the same period to period. This is because we are calculating its steady-state value, and in steady state the variables are constant period to period. In the previous exercises in which the optimal consumption path was calculated, the slope of the same depended on the relationship between the discount factor and the interest rate, which was assumed exogenous. However, in this general equilibrium model, the interest rate is an endogenous variable, and its equilibrium value is such that, given a discount factor, it makes the rest of the variables constant, so the optimal path of the resulting consumption is horizontal. In fact, we can verify that in a steady state, given a value of $\beta = 0.96$, corresponding to an intertemporal subjective rate of $\theta = 0.042$, the steady-state value of the interest rate is 10.2% per period. Discounting the physical depreciation rate of capital, which is 6% per period, it turns out that the net return of capital is 0.102-0.06=0.042, which is exactly equal to the value of the subjective rate of intertemporal preference.

Second, the model can be solved directly in Excel using the log-linearized system. The numerical resolution of the log-linearized model corresponds to the spreadsheet "RBC-2.xlsx". First, we define the parameters of the model. In this exercise, we will use the same parameters as in the previous exercise – the intertemporal discount factor, the elasticity of production with respect to the stock of capital and the rate of depreciation of capital. We also calculate two parameters that are a combination of the previous ones to simplify the used expressions. The corresponding values appear in cells "B14" to "B18". In column "C", these values are reproduced, in order to perform a sensitivity analysis and study the implications of changes in the values of these parameters. Below, we present the steady-state values, rows 21 to 25, which are the same as those obtained in the previous exercise. In column "B", the steady-state values are presented with the initial values, while in column "C", these values are presented with the final values. Given that in this exercise we have considered TFP as an exogenous variable, we have introduced its initial value in cell "B25". If we want to introduce a new value to simulate a permanent technological disturbance, we would do so by changing the corresponding value in cell "C25".

Rows 28 and 29 calculate the eigenvalues associated with the dynamic system, in column "B" for the initial steady state and in column "C" for the final steady state. Given the restrictions on the parameters, the roots are going to be real, so the calculation of the imaginary part is not necessary. If we place the cursor in cell "B28", the expression that appears is:

```
=((-((Alpha_0-1)*OMEGA_0*PHI_0-Alpha_0*(1-Beta_0)))/(Alpha_0*Beta_0)
-ROOT((((Alpha_0-1)*OMEGA_0*PHI_0-Alpha_0*(1-Beta_0))/(Alpha_0*Beta_0))^2
      -4*((Alpha_0-1)*OMEGA_0*PHI_0)/(Alpha_0*Beta_0)))/2
```

which corresponds to the first root, while in cell "B29", the equivalent expression for the other root appears. In rows 32 and 33, the module of each root plus the unit is calculated.

The information that results from numerically simulating this model appears in the "G-U" columns. Column "G" is the time index. The variables of the model are defined in the columns "H", the consumption, "I" corresponds to the investment, "J" the level of production, "K" the stock of capital, while the columns "L", "M", "N", and "O", present the previous variables in the same order, but in logarithms. Next, the column "P" corresponds to the logarithmic deviation of the consumption with respect to its steady-state value, the "Q" is the logarithmic deviation of the investment with respect to its steady-state value, which is simply the difference between what is produced and what is consumed, in terms of deviations. Column "R" is the logarithmic deviation of production, which depends on the deviation of the stock of capital from its steady-state value, and column "S" is the logarithmic deviation of the capital stock with respect to its steady-state value.

To determine the initial consumption, column "H", we will start from its steady-state value. To determine the consumption in period 1 (i.e., "H4"), we use the following expression:

```
=EXP(P4+LN(Css_1))
```

This same expression appears in the following cells. A similar expression appears in the "K" column to calculate the stock of capital from its logarithmic deviation. To determine the values corresponding to column "J", we use the expression, "`=PTF_0*K_0^Alpha_0`", corresponding to the initial period, while the initial expression entered in the column "I" is "`=K_0*Delta_0`". For the following periods, it is determined using the following expression, "`=Alpha1*Q4`". The only value that would change is "Q" that corresponds to the logarithmic variations of the stock of capital in each period of time with respect to the steady state.

Columns "P" to "S" show the deviations of each variable with respect to its steady state, where the key cells are "P4" and "S4". The "P" column corresponds to the logarithmic variation in consumption with respect to its steady-state value. For the initial period (zero), cell "P3", is the difference between the logarithm of the steady-state consumption and the logarithm of the same, whose result is zero. Cell "P4" contains the new value of the deviation of the consumption before a disturbance that places said variable in the new stable path. Thus, $\widehat{c}_1$ is determined with the following expression:

```
=Alpha_1*(Alpha_1-1)*OMEGA_1/((Alpha_1-1)*OMEGA_1*PHI_1
            +Alpha_1*Beta_1*Lambda1_1)*S4
```

which corresponds to jump value. For the successive periods, the consumption deviation is determined using the expression, "=N4+R4", that is, the consumption in the previous period plus the variation in consumption, which is the value corresponding to cell "R4". This expression is copied into the remaining rows of that column.

Column "S" contains the differences of the logarithm of the stock of capital with respect to the steady state. For the period zero it would be determined using the following expression, "=LN(K3)-LN($K$3)", it is the difference between the logarithm of the stock of capital in steady state and the same, therefore it is zero. On the other hand, to calculate this deviation in period 1, corresponding to cell "S4", it would correspond to the difference between the logarithm of the stock of capital in the steady state and the logarithm of the stock of capital in the final steady state, the expression that we use is:

```
=LN(K_0)-LN(Kss_1)
```

On the other hand, the column "R" calculates the deviations of the production level, using the expression "=Alpha_0*S3". In the following periods and until the end of the column, the expression used is, "=Alpha_1*S4".

Finally, columns "T" and "U" show the variations in the deviations of consumption and capital stock, respectively. Column "T" contains the variations of the logarithmic deviations of the consumption with respect to the steady state. If we place the courses in cell "T3", the expression that appears is:

```
=-(Alpha_0-1)*OMEGA_0*PHI_0/(Alpha_0*Beta_0)*P3
        +(Alpha_0-1)*OMEGA_0/Beta_0*S3
```

This same expression appears in the following cells of this column but referred to the values of the final parameters. Finally, column "U" presents the value of the variations of the deviations of the stock of capital with respect to the initial steady state. In this case, the expression that appears in cell "U3" is

```
=-PHI_0/(Alpha_0*Beta_0)*P3+(1-Beta_0)/Beta_0*S3
```

This same expression appears in the following cells of the column but refers to the values of the parameters in the final steady state.

3 Exercises

1. Study the effects of a change in the elasticity of output with respect to capital, α (for example, the value changes from 0.35 to 0.3). Analyze the consequences of this change on the steady state and on the effect of a technological shock. Repeat this exercise using the spreadsheets "RBC-1.xlsx" and "RBC-2.xlsx".

2. Suppose that an earthquake decreases by 20% the capital stock of the economy. Using the spreadsheet "RBC-2.xlsx", The Model tab, study what the effects of this shock. To perform this experiment, simply enter in the spreadsheet that the initial value of the capital stock in cell "K3" is 20% lower than the corresponding steady-state value, i.e., `"K_0*0.8"`.

3. Solve the dynamic general equilibrium model assuming that households' utility function is:

$$U(C_t, L_t) = \gamma \ln C_t + (1 - \gamma) \ln(1 - L_t)$$

   where $0 < \gamma < 1$. Build a spreadsheet similar to the "RBC-1.xls", which calculates both the optimal path of consumption and the optimal labor supply. What effects does a positive technological disruption have on the labor supply?