The Ramsey optimal growth model

1 The model

The standard Dynamic General Equilibrium model that is currently used in macroeconomic analysis has as its starting point the model developed by Frank Plumpton Ramsey (1903-1930) almost a century ago. The Ramsey model, also called the optimal growth model (as opposed to the Solow-Swan model, in which there is no optimality criterion and in which the saving rate is exogenous), or named the Ramsey-Cass-Koopmans model, has become the theoretical framework of reference of modern macroeconomic analysis, not only to study the behavior of the economy in the long-run but also for the study of economic fluctuations in the short-run.

The structure of the model is similar to RBC model. Thus, we assume the existence of two agents: households and firm, which interact in a competitive environment. Households aim to maximize the discounted sum of their utility throughout their life cycle, while the firm's goal is to maximize profits period-by-period. The model has a new exogenous variable: population growth, since demographic dynamics is a key factor to take into account when studying economic growth. We can think in the household as a family where the number of members increases over time. Secondly, we define all the variables in per capita terms, since it is the relevant measure in this context.

Families In Ramsey's model, we introduce the concept of family and the variation in the number of family members over time. In particular, we assume that the individuals that inhabit an economy belong to the same family. This is a natural way of introducing the concept of infinite life for consumers that is commonly used in micro-based macro models. In this way, individuals can have finite life and cease to exist at a particular moment in time, but the family is immortal. Using the concept of family has important connotations from the economic point of view, since we are referring to the existence of a relationship of kinship between the individuals that live in an economy, so it is to be assumed that the welfare of the future generations also affects the welfare of agents at the present time. In other words, the individuals of a generation would also be concerned with the welfare of individuals of future generations, which results in the decision-maker, the family, acting as an agent with infinite life.

Population, L_t, is defined as:

$$L_t = L_{t-1}(1+n) = L_0(1+n)^t \tag{1}$$

where $n > 0$ is the population growth rate, and L_0 the initial population which is normalized to one. We defined all the variables in terms of per capita (denoted by lowercase), simply dividing by population. Therefore, per capita consumption would be defined as $c_t = C_t/L_t$, and so on.

Assuming a logarithmic utility function, the problem to be maximized by the family is:

$$\max_{\{c_t\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \left(\frac{1+n}{1+\theta}\right)^t \ln c_t \tag{2}$$

subject to the budget constraint,

$$c_t + (1+n)k_{t+1} = W_t + (R_t + 1 - \delta)k_t \tag{3}$$

given $k_0 > 0$, where $\beta \in (0,1)$ is the discount factor defined as $\beta = 1/(1+\theta)$, being $\theta > 0$, the intertemporal subjective preference rate. We have to impose this additional condition that $\theta > n$. We assume the existence of a perfect foresight, so we can directly eliminate the mathematical expectation operator.

The Lagrange auxiliary function would be:

$$\mathcal{L} = \sum_{t=0}^{T} \left(\frac{1+n}{1+\theta}\right)^t \ln c_t - \lambda_t \left[c_t + (1+n)k_{t+1} - W_t - (R_t + 1 - \delta)k_t\right] \tag{4}$$

First-order conditions, for t=0,1,2,. . . ,T, are the following:

$$\frac{\partial \mathcal{L}}{\partial c_t} : \left(\frac{1+n}{1+\theta}\right)^t \frac{1}{c_t} - \lambda_t = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} : \lambda_{t+1}(R_{t+1} + 1 - \delta) - \lambda_t(1+n) = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} : c_t + k_{t+1} - W_t - \frac{(R_t + 1 - \delta)}{(1+n)}k_t = 0 \tag{7}$$

From the first-order conditions we obtain the following optimal path for per capita consumption,

$$c_{t+1} = \beta(R_{t+1} + 1 - \delta)c_t \tag{8}$$

The firms Next, we will analyze the behavior of firms. Since our variables of interest are defined in per capita terms, and also the household maximization problem had been solved in per capita terms, we also have to define the firm profit maximization problem in per capita terms. Firms maximize the following per capita profit function:

$$\max \pi_t = A_t k_t^{\alpha} - w_t - R_t k_t \tag{9}$$

The first-order condition from the maximization is given by:

$$\frac{\partial \pi}{\partial k} = \alpha A_t k_t^{\alpha-1} - R_t = 0 \tag{10}$$

From the previous expression, we obtain that the marginal productivity of capital per capita is equal to the interest rate:

$$\alpha A_t k_t^{\alpha-1} = R_t \tag{11}$$

Given that the maximization of profits is carried out in per capita terms, we do not have a first-order condition with respect to employment. In this case, the equilibrium wage would be obtained as the difference between total income and capital income, in per capita terms,

$$W_t = A_t k_t^{\alpha} - k_t \alpha A_t k_t^{\alpha-1} = A_t k_t^{\alpha} - \alpha A_t k_t^{\alpha} = (1-\alpha) y_t \tag{12}$$

The competitive equilibrium The competitive equilibrium is given by the vector of values for the endogenous variables in which the optimal plans of the family and the firm coincide. That is, it is defined by that situation in which the stock of capital per capita that firms want to hire is equal to the amount of financial assets per capita that households decide to accumulate. To obtain the expressions corresponding to the equilibrium of the model, we have to substitute the price of the productive factors (wage and interest rate) in the expressions that reflect the optimal decisions of the households. Thus, we obtain a system of two difference equations, which allow us to determine both the per capita consumption and the capital stock per capita.

$$c_{t+1} = \beta(\alpha A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta)c_t \tag{13}$$

$$c_t + (1+n)k_{t+1} = A_t k_t^{\alpha} + (1-\alpha)k_t \tag{14}$$

Steady state The steady state of the economy is defined by the following expressions:

$$\overline{k} = \left(\frac{1-\beta+\beta\delta}{\alpha A\beta}\right)^{\frac{1}{\alpha-1}} \tag{15}$$

$$\overline{y} = A\left(\frac{1-\beta+\beta\delta}{\alpha A\beta}\right)^{\frac{1}{\alpha-1}} \tag{16}$$

$$\overline{c} = A\left(\frac{1-\beta+\beta\delta}{\alpha A\beta}\right)^{\frac{\alpha}{\alpha-1}} - (n+\delta)\left(\frac{1-\beta+\beta\delta}{\alpha A\beta}\right)^{\frac{1}{\alpha-1}} \tag{17}$$

$$\overline{i} = (n+\delta)\left(\frac{1-\beta+\beta\delta}{\alpha A\beta}\right)^{\frac{1}{\alpha-1}} \tag{18}$$

$$\overline{c} = A\left(\frac{1-\beta+\beta\delta}{\alpha A\beta}\right)^{\frac{\alpha}{\alpha-1}} - (n+\delta)\left(\frac{1-\beta+\beta\delta}{\alpha A\beta}\right)^{\frac{1}{\alpha-1}} \tag{19}$$

Log-linearized model The log-linear equations of the model are give by,

$$\widehat{y}_t = \alpha\widehat{k}_t \tag{20}$$

$$\widehat{k}_{t+1} = -\left[\frac{1-\beta+\beta\delta}{\alpha\beta(1+n)} - \frac{(\delta+n)}{(1+n)}\right]\widehat{c}_t + \frac{1}{\beta(1+n)}\widehat{k}_t \tag{21}$$

$$\widehat{c}_{t+1} = \widehat{c}_t + (1-\beta+\beta\delta)(\alpha-1)\widehat{k}_{t+1} \tag{22}$$

$$\widehat{i}_t = \frac{1-\beta+\beta\delta}{\alpha\beta(n+\delta)}\widehat{y}_t - \frac{1-\beta+\beta\delta-\alpha\beta(n+\delta)}{\alpha\beta(n+\delta)}\widehat{c}_t \tag{23}$$

where $\widehat{x}_t = \ln X_t - \ln\overline{X}$.

Solution The log-linearized model can be reduced to a system of two linear difference equations, such as,

$$\begin{bmatrix} \Delta\widehat{c}_t \\ \Delta\widehat{k}_t \end{bmatrix} = \begin{bmatrix} -\frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)} & \frac{(\alpha-1)\Omega}{\beta(1+n)} \\ -\frac{\Gamma}{\alpha\beta(1+n)} & \frac{1-\beta(1+n)}{\beta(1+n)} \end{bmatrix} \begin{bmatrix} \widehat{c}_t \\ \widehat{k}_t \end{bmatrix} \tag{24}$$

where

$$\Omega = 1-\beta+\beta\delta \tag{25}$$

$$\Gamma = 1-\beta+\beta\delta-\alpha\beta(\delta+n) \tag{26}$$

Eigenvalues Next, we perform the stability analysis to calculate the value of the eigenvalues associated with this model. This model will have a saddle-point solution, which implies that one of the roots will be positive and the other will be negative. To calculate the value of the eigenvalues associated with this model, we solve

$$Det \begin{bmatrix} -\frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)} - \lambda & \frac{(\alpha-1)\Omega}{\beta(1+n)} \\ -\frac{\Gamma}{\alpha\beta(1+n)} & \frac{1-\beta(1+n)}{\beta(1+n)} - \lambda \end{bmatrix} = 0 \tag{27}$$

giving rise to the following equation of the second degree:

$$\lambda^2 + \left[\frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)} - \frac{1-\beta(1+n)}{\beta(1+n)}\right]\lambda + \tag{28}$$

$$\frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)\beta(1+n)} - \left(\frac{1-\beta(1+n)}{\beta(1+n)}\right)\frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)} = 0 \tag{29}$$

or equivalently

$$\lambda^2 + \frac{(\alpha-1)\Omega\Gamma - \alpha + \alpha\beta(1+n)}{\alpha\beta(1+n)}\lambda + \frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)} = 0 \tag{30}$$

Soving, we find that eigenvalues of the log-linearized solution are given by,

$$\lambda_{1,2} = \frac{-\left[\frac{(\alpha-1)\Omega\Gamma-\alpha+\alpha\beta(1+n)}{\alpha\beta(1+n)}\right] \pm \sqrt{\left(\frac{(\alpha-1)\Omega\Gamma-\alpha+\alpha\beta(1+n)}{\alpha\beta(1+n)}\right)^2 - 4\frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)}}}{2} \tag{31}$$

Jump forward-looking variable In the case of the Ramsey model, we are solving the model in terms of two variables: per capita consumption and the stock of capital per capita. While consumption is a forward-looking variable, which can be adjusted instantaneously in a new economic environment (is a flexible variable), the stock of capital is a rigid variable, since it implies the change in a physical variable that cannot be realized instantaneously, but rather needs time for its adjustment. To calculate the adjustment that has to occur in per capita consumption, we start from its dynamic equation:

$$\Delta\widehat{c}_t = \frac{(\alpha-1)\Omega}{\beta(1+n)}\widehat{k}_t - \frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)}\widehat{c}_t \tag{32}$$

On the other hand, the stable path would be defined by the following trajectory,

$$\Delta\widehat{c}_t = \lambda_1 \widehat{c}_t \tag{33}$$

Matching both expressions results in:

$$\frac{(\alpha-1)\Omega}{\beta(1+n)}\widehat{k}_t - \frac{(\alpha-1)\Omega\Gamma}{\alpha\beta(1+n)}\widehat{c}_t = \lambda_1 \widehat{c}_t \tag{34}$$

Solving, we find that the value of the jumping variable (the forward-looking variable, i.e., consumption) in the instant when a shock hits the economy is given by,

$$\widehat{c}_t = \frac{\alpha(\alpha-1)\Omega}{(\alpha-1)\Omega\Gamma + \alpha\beta(1+n)\lambda_1}\widehat{k}_t \tag{35}$$

where λ_1 is the stable eigenvalue, in order the economy be at the stable path to the new steady state.

2 Taking the model to Excel

The model is solved in the spreadsheet "Ramsey.xlsx". In cells "B4" to "B8" appear the calibrated values of the parameters that define the initial steady state: the intertemporal discount rate, "`Beta_0`", the technological parameter of the stock of capital, "`Alpha_0`" and the depreciation physical rate of capital, "`Delta_0`". We also calculate the value of the combinations of parameters that will be useful to simplify the expressions in cells "B7" and "B8", which we have named "`OMEGA_0" and "GAMMA_0`". In column "C", these same parameters appear, in order to be able to perform direct simulations of changes in their value (sensitivity analysis). Initially, these values are the same in the initial state as in the final state, then the value of the exogenous variables is determined in cell "B11" for total factor productivity and cell "B12" for the population growth rate. The name assigned to these cells is "`PTF_0`" and "`n_0`", respectively. The same as the case of the parameters in column "B", the values of the initial steady state are entered and in column "C" the new values in the case in which you want to perform a disturbance analysis. Then, in cells "B15" to "B18", we have the values of the initial steady state for the stock of capital, production, consumption and investment. If we place the cursor in cell "B15", the expression we find is:

```
=(OMEGA_0/(Alpha_0*PTF_0*Beta_0))^(1/(Alpha_0-1))
```

expression that corresponds to the steady-state value for the stock of capital. Cell "B16" contains the steady-state value of the production level, through the expression:

```
=PTF_0*kss_0^Alpha_0
```

which is simply the aggregate production function of the economy. If we place the cursor in cell "B17", we obtain the steady-state value for consumption, which we calculate using the following expression:

```
=yss_0-(n_0+Delta_0)*kss_0
```

Finally, cell "B18" contains the steady-state value of the investment, which we simply calculate as the difference between production and consumption. In cells "C15" to "C18", we obtain the steady-state values for these variables in the final situation, whose expressions are equivalent to the previous ones but are referenced to the values of the parameters at the final moment.

The eigenvalues associated with this system are calculated in cells "B21" and "B22". If we place the cursor in cell "B21", the expression that appears is:

```
=(-(((Alpha_0-1)*OMEGA_0*GAMMA_0-Alpha_0+Alpha_0*Beta_0*(1+n_0))
                    /(Alpha_0*Beta_0*(1+n_0)))
          -ROOT((((Alpha_0-1)*OMEGA_0*GAMMA_0-Alpha_0
     +Alpha_0*Beta_0*(1+n_0))/(Alpha_0*Beta_0*(1+n_0)))^2
  -4*(Alpha_0-1)*OMEGA_0*GAMMA_0/(Alpha_0*Beta_0*(1+n_0))))/2
```

which corresponds to the negative (stable) eigenvalue. We find an equivalent expression, but for the positive root, in cell "B22". These eigenvalues correspond to the parameters that define the initial steady state. Cells "C21" and "C22" calculate the eigenvalues corresponding to the parameters that determine the final steady state. Finally, cells "B25" and "B26" show the module of each eigenvalue plus the unit, in order to determine the stability of the system.

The variables of the model are calculated in columns "E" to "S". In column "E", the time index is shown. In columns "F" to "I", the value of the variables in levels is calculated. The values of these variables in period 0 correspond to the initial steady state. In column "F", we have calculated the consumption for each period. If we place the cursor in cell "F3", the expression that appears is "`=css_0`", which is the consumption in the initial steady state. The expression that appears in cell "F4" is

```
=EXP(N4+LN(css_1))
```

Columns "J" to "M" simply calculate the logarithm of the variables, while columns "N" to "Q" calculate the deviation of each variable with respect to its steady-state value. The key cells to numerically solve the model are the "N4" and the "Q4". If we place the cursor in cell "N4", the expression that appears is

```
=(Alpha_1*(Alpha_1-1)*OMEGA_1/((Alpha_1-1)*OMEGA_1*GAMMA_1
          +Alpha_1*Beta_1*(1+n_1)*Lambda1_1))*Q4
```

that calculates the jump that per capita consumption must take to reach the stable path. The rest of the cells in this column are simply calculated as the consumption value in the previous period plus the change in that period. Thus, for example, the expression in cell "N5" is "`=N4+R4`". For its part, cell "Q4" introduces the deviation that occurs in the stock of capital per capita with respect to its new steady-state value, so the expression corresponding to this cell is

```
=LN(kss_0)-LN(kss_1)
```

Finally, in the columns "R" and "S", the variations in the variables are calculated. The expression entered in cell "R3" is

```
=-((Alpha_0-1)*OMEGA_0*GAMMA_0/(Alpha_0*Beta_0*(1+n_0)))*N3
         +((Alpha_0-1)*OMEGA_0/(Beta_0*(1+n_0)))*Q3
```

expression that corresponds to the dynamic equation of per capita consumption. This same expression appears in the following cells, but referred to the values of the parameters and exogenous variables in the new steady state. The expression entered in cell "S3" is

```
=-GAMMA_0/(Alpha_0*Beta_0*(1+n_0))*N3
+((1-Beta_0*(1+n_0))/(Beta_0*(1+n_0)))*Q3
```

expression that corresponds to the dynamic equation of the stock of capital per capita.

3 Exercises

1. Starting from the calibrated model in the text, suppose there is a decrease in the population growth rate to 0.01. Using the spreadsheet "Ramsey.xlsx", The Model tab, study the effects of this shock (you only need to change the value of cell "C12"). Why is the adjustment of the economy to this disturbance instantaneous? What consequences does it have on the level of per capita consumption and the stock of capital per capita?

2. Suppose that there is an increase in the value of the parameter α, which means that the proportion of capital income increases over total income (thus decreasing the proportion of labor income). Suppose a new value of 0.4. What effects does this new value of this parameter have on the steady state of the economy and the dynamics towards it?

3. From the calibration of the Ramsey model made in the spreadsheet "Ramsey.xlsx", determine what is the steady state savings rate. Compare this savings rate with the technological parameter that determines the elasticity of the production level with respect to the stock of capital. What conclusions are drawn regarding the golden rule?