The Tobin Q model

1 The model

Firms, together with households, make up the two main economic agents which determine the behavior of an economy through in micro-founded macroeconomic models. Firms represent the productive sector of the economy, that is, they are the economic agents that produce the final goods that are consumed or invested, by taking decisions about the demand of productive factors.

In studying optimal decisions by the firms, we assume that their objective function is profits maximization, subject to the technological restriction. The key variable we analyze here is the investment decision by the firm. Thus, we assume that firms own their physical capital and, therefore, are the ones that make investment decisions using households savings as a source of financing. By using this alternative specification, we can separate the saving decision from the investment decision and thus, obtain the investment demand in a dynamic environment, since the investment decision today, will affect the future flow of profits of the firms, in a way equivalent to how consumers determine their level of savings today and their future level of consumption.

The reference model used to analyze the determinants of investment is the so-called Tobin-Q model, developed by James Tobin in the late 1960s. In this model the optimal investment rate depends on a ratio, called the Tobin's Q, defined as the ratio of the firm's market value to the replacement cost of installed capital. That is, the ratio compares what the firm is valued with respect to what it would cost to install again all the capital available to the firm, which is equivalent to comparing the profitability of an investment with its cost. In this context, the investment will be a function of the value of this ratio and, in this way, any factor that affects this ratio will also affect the investment decision.

The Tobin Q model is as follows. We assume a Cobb-Douglas production function:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha} \tag{1}$$

where Y_t is output, K_t is capital and L_t is labor. The parameter α represents the elasticity of output with respect to capital. Labor is normalized to 1.

Given the assumption that the firm is the owner of capital, we use the following definition for profits, Π_t:

$$\Pi_t = Y_t - W_t L_t - I_t - C(I_t, K_t) \tag{2}$$

where W_t is wage, I_t is investment and $C(I_t, K_t)$ is an investment adjustment cost function. Capital accumulation process is given by:

$$K_{t+1} = (1-\delta)K_t + I_t \tag{3}$$

where δ is the capital depreciation rate. The investment adjustment cost function is given by:

$$C(I_t, K_t) = \frac{\phi}{2}\left(\frac{I_t - \delta K_t}{K_t}\right)^2 K_t \tag{4}$$

where ϕ is a parameter representing how adjustment cost reduces profits when capital changes. The problem for the firm consist in solving the following profits maximization problem:

$$\max \sum_{t=0}^{T} \frac{1}{(1+R_t)^t}\left[Y_t - W_t - I_t - C(I_t, K_t)\right] \tag{5}$$

subject to the technology, the capital accumulation process and the adjustment costs. By substituting the technological constraint and the adjustment costs in the objective function we obtain the following auxiliary function of Lagrange:

$$\begin{aligned} V &= \sum_{t=0}^{T} \frac{1}{(1+R_t)^t}\left[K_t^{\alpha} - W_t - I_t - \frac{\phi}{2}\left(\frac{I_t - \delta K_t}{K_t}\right)^2 K_t\right] \\ &\quad -\lambda_t(K_{t+1} - I_t - (1-\delta)K_t) \end{aligned} \tag{6}$$

The first-order conditions to the previous problem are the following:

$$\frac{\partial V}{\partial K_{t+1}} : \frac{1}{(1+R_{t+1})^{t+1}} \left[\alpha K_{t+1}^{\alpha-1} - \frac{\phi}{2} \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right)^2 + \phi \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}} \right]$$
$$+ \lambda_{t+1}(1-\delta) - \lambda_t = 0 \tag{7}$$
$$\frac{\partial V}{\partial I_t} : -\frac{1 + \phi \left(\frac{I_t - \delta K_t}{K_t} \right)}{(1+R_t)^t} + \lambda_t = 0 \tag{8}$$
$$\frac{\partial V}{\partial \lambda_t} : -K_{t+1} + I_t + (1-\delta) K_t = 0 \tag{9}$$

Operating we find that:

$$\alpha K_{t+1}^{\alpha-1} - \frac{\phi}{2} \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right)^2 + \phi \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}} =$$
$$(1+R_t) \left[1 + \phi \left(\frac{I_t - \delta K_t}{K_t} \right) \right] - \left[1 + \phi \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right) \right] (1-\delta) \tag{10}$$

Finally, we define Tobin's Q ratio. This ratio is defined as the market value of the firm with respect to the replacement cost of installed capital. In our case, we define this ratio in marginal terms, denoted as q. That is, the q ratio would be the variation in the market value of the firms with respect to the variation in the replacement cost of capital, that is, the cost of investing an additional unit. Under certain assumptions, the q ratio is equal to the average of the Q ratio. The q ratio is defined as:

$$q_t = \lambda_t (1+R_t)^t \tag{11}$$

By using the definition of the Lagrange parameter obtained above, it turns out that,

$$q_t = 1 + \phi \left(\frac{I_t - \delta K_t}{K_t} \right) \tag{12}$$

$$\begin{aligned} (1-\delta) q_{t+1} &= (1+R_t) q_t - \alpha K_{t+1}^{\alpha-1} + \frac{\phi}{2} \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right)^2 \\ &\quad - \phi \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}} \end{aligned} \tag{13}$$

From the above two equations we obtain that:

$$\Delta K_t = (q_t - 1)\frac{K_t}{\phi} \tag{14}$$

$$\Delta q_t = \frac{(R_t + \delta)q_t - \alpha K_{t+1}^{\alpha-1} + \frac{\phi}{2}\left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}}\right)^2 - \phi\left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}}{(1-\delta)} \tag{15}$$

where $\Delta q_t = q_{t+1} - q_t$, and where $\Delta K_t = K_{t+1} - K_t$. However, this is a system of non-linear equations. To obtain a solution, the equations must be linearized.

Steady State In steady state, the capital accumulation equation is:

$$\Delta K = (\overline{q} - 1)\frac{\overline{K}}{\phi} = 0 \tag{16}$$

and hence:

$$\overline{q} = 1 \tag{17}$$

For the dynamic equation for q,

$$\Delta q = \frac{(R + \delta)\overline{q} - \alpha\overline{K}^{\alpha-1}}{(1-\delta)} = 0 \tag{18}$$

resulting that:

$$\overline{K} = \left(\frac{R+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{19}$$

Log-linearized model To obtain the log-linearization of our system of equations, we will express the variables of the model as the log-linear deviation with respect to its steady-state values. The log-linear deviation of a variable, x_t, with respect to its steady-state value, $\overline{x}_t$, we will define as $\widehat{x}_t$, where $\widehat{x}_t = \ln x_t - \ln \overline{x}_t$. To construct the equations in log-linear form, we will use the following three basic rules:

Each one of the variables can be defined as:

$$x_t \approx \overline{x}_t \exp(\widehat{x}_t) \approx \overline{x}_t(1 + \widehat{x}_t) \tag{20}$$

When two variables are multiplying, then:

$$x_t z_t \approx \overline{x}_t(1+\widehat{x}_t)\overline{z}_t(1+\widehat{z}_t) \approx \overline{x}_t\overline{z}_t(1+\widehat{x}_t+\widehat{z}_t) \tag{21}$$

that is, we assume that the product of two deviations with respect to its steady states, $\widehat{x}_t\widehat{z}_t$, is a very small number and approximately equal to zero.

The third rule refers to the powers, such that:

$$x_t^a \approx \overline{x}_t^a(1+\widehat{x}_t)^a \approx \overline{x}_t^a(1+a\widehat{x}_t) \tag{22}$$

Using the above rules, the log-linear approximation to the model is given by,

$$\widehat{k}_{t+1} = \widehat{k}_t + \frac{1}{\phi}\widehat{q}_t \tag{23}$$

$$\widehat{q}_{t+1} = (1+R_t)\widehat{q}_t - (\alpha-1)(R_t+\delta)\left(\widehat{k}_t + \frac{1}{\phi}\widehat{q}_t\right) \tag{24}$$

Solution The log-linearized model can be represented by the following system of linear difference equations:

$$\begin{bmatrix} \Delta\widehat{q}_t \\ \Delta\widehat{k}_t \end{bmatrix} = \begin{bmatrix} \frac{R_t\phi-(\alpha-1)(R_t+\delta)}{\phi} & -(\alpha-1)(R_t+\delta) \\ \frac{1}{\phi} & 0 \end{bmatrix} \begin{bmatrix} \widehat{q}_t \\ \widehat{k}_t \end{bmatrix} \tag{25}$$

where $\widehat{x}_t = \ln X_t - \ln \overline{X}$.

Eigenvalues Once the model is linearized, we can then calculate the roots (eigenvalues) associated with the coefficient matrix to study the stability conditions of the dynamic system. For this, we calculate:

$$Det \begin{bmatrix} \frac{R_t\phi-(\alpha-1)(R_t+\delta)}{\phi} - \lambda & -(\alpha-1)(R_t+\delta) \\ \frac{1}{\phi} & 0-\lambda \end{bmatrix} = 0 \tag{26}$$

The corresponding second order equation is:

$$\lambda^2 - \frac{R_t\phi-(\alpha-1)(R_t+\delta)}{\phi}\lambda + \frac{(\alpha-1)(R_t+\delta)}{\phi} = 0 \tag{27}$$

The solution of the system is a saddle point. Eigenvalues are,

$$\lambda_{1,}\lambda_2 = \frac{\frac{R_t\phi-(\alpha-1)(R_t+\delta)}{\phi} \pm \sqrt{\left(\frac{R_t\phi-(\alpha-1)(R_t+\delta)}{\phi}\right)^2 - 4\frac{(\alpha-1)(R_t+\delta)}{\phi}}}{2} \tag{28}$$

Jump forward-looking variable adjustment The procedure to calculate the readjustment in expectations is similar to the one done previously. For this, we start with the equation that describes the dynamic behavior of the q ratio with respect to its steady state:

$$\Delta\widehat{q}_t = \frac{R_t\phi - (\alpha - 1)(R_t + \delta)}{\phi}\widehat{q}_t - (\alpha - 1)(R_t + \delta)\widehat{k}_t \tag{29}$$

Equivalently, we can define the following stable trajectory:

$$\Delta\widehat{q}_t = \lambda_1\widehat{q}_t \tag{30}$$

Matching both expressions at the time the disturbance occurs ($t = 1$) results in:

$$\frac{R_t\phi - (\alpha - 1)(R_t + \delta)}{\phi}\widehat{q}_1 - (\alpha - 1)(R_t + \delta)\widehat{k}_1 = \lambda_1\widehat{q}_1 \tag{31}$$

Solving for the value of the jumping variable (the forward-looking variable, i.e., the q) in the instant when a shock hits the economy yields,

$$\widehat{q}_t = \frac{(\alpha - 1)(R_t + \delta)}{\frac{R_t\phi - (\alpha - 1)(R_t + \delta)}{\phi} - \lambda_1}\widehat{k}_t \tag{32}$$

where λ_1 is the stable eigenvalue, in order the economy to move to the stable saddle path corresponding to the new steady state after a shock.

2 Taking the model to Excel

The model is solved in the spreadsheet named "Tobin.xlsx", The Model tab. We have calculated three types of endogenous variables: variables in levels (q, K), variables in terms of logarithmic deviations with respect to the steady state $(\widehat{q},\widehat{k})$, and time variations of the logarithmic deviations with respect to the steady state, $(\Delta\widehat{q},\Delta\widehat{k})$.

Cells "B12-B14" show the initial values of the parameters. In cell "B12" the initial value of the parameter is presented, cell that we call "`Alpha_0`". We also include in cell "C12" the final value for this parameter, to be used in the sensitivity analysis, in case we are interested in studying the behavior of the model before a change in the value of this parameter. We call this final value "`Alpha_1`". Similarly, cell "B13" shows the value of the depreciation rate of the initial capital (called "`Delta_0`"), while in cell "B14" the value assigned to the parameter of the adjustment cost function is presented (which we have called "`Phi_0`"). Next, we show the values of the exogenous variable, which in the case of this model is the interest rate, both in the initial moment and in the final moment, in order to perform perturbation analysis. These values appear in cells "B17" and "C17" and their final values in cells "C13" and "C14", cells that we call "`R_0`" and "`R_1`".

The steady-state values, both with the initial values of the parameters and exogenous variable, and with the final values appear in cells "B20", "C20", "B21" and "C21". These cells are called "`qbar_0`" and "`qbar_1`" for the initial and final steady states of the q ratio, and "`kbar_0`" and "`kbar_1`", for the initial and final steady states of the capital stock. As initially, both the final and initial values of parameters and exogenous variables are the same, also the steady state shows the same values in both situations. Finally, the eigenvalues appear in cells "B24" and "B25" for the initial situation and in cells "C24" and "C25" for the final situation, while the module of the eigenvalues plus the unit are calculated in cells "B28" and "B29" for the initial situation and "C28" and "C29" for the final.

Below are the columns where we will calculate the variables of the model. In column "H", the time index is included. Columns "I" to "N" show the values of the different variables. Column "I" shows the values of the q ratio. If we place the cursor in cell "I3", we see that the expression "`=qbar_0`" appears, since we start from the initial steady state. If we place the cursor in cell "I4", the expression that appears is "`=EXP(K4+LN(qbar_1))`", indicating that in the next period the value of the q ratio is the value of its deviation from the steady state, values that appear in the "K" column, plus the new steady state for the q ratio. This is because the deviations from the steady state are defined as $\widehat{q}_t = q_t - \overline{q}$, so the value of the variable can be calculated as $q_t = \widehat{q}_t - \overline{q}$.

This same expression appears in the following cells in this column. Next, column "J" shows the values of the capital stock. Again, if we place the cursor in cell "J3", the expression that appears is "`=kbar_0`", which corresponds to the value in the initial steady state of the capital stock. If we place the cursor in cell "J4", the expression that appears is "`=EXP(L4+LN(kbar_1))`". This expression calculates the stock of capital in this period, such as cell "L4", in which the deviation of the stock of capital appears with respect to its steady state, plus the logarithm of value of the new steady state for the stock of capital.

The "K" and "L" columns show the deviations with respect to the steady state of the q ratio and capital stock, respectively. Cell "K5" (and the following) of column "K" includes the expression "=K4+M4", in which we calculate the value of the variable as its value in the previous period plus the change produced in it. In cell "L3" we find the expression:

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=LN(J3)-LN(kbar_0)
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that is, the difference between the value of the stock of capital at the moment in which the disturbance occurs, which is equal to its initial steady state and the new steady-state value. Cell "L4" and the following calculate the value of the variable as the value of the previous period plus the change produced in the variable.

The two key cells in the analysis of this model are "K4" and "L4". In cell "K4", the expression is,

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=((Alpha_1-1)*(R_1+Delta_1)*Phi_1/(R_1*Phi_1-(Alpha_1-1)
*(R_1+Delta_1)-Phi_1*Lambda1_1))*(LN(kbar_0)-LN(kbar_1))
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This cell contains the readjustment of the q ratio once a disturbance occurs. As we have indicated previously, the dynamics of this model are determined by a saddle point, which means that only some of the trajectories are convergent towards the steady state. These trajectories converge to the so-called stable path, to which the q variable moves instantaneously once a disturbance has occurred. This instantaneous change is due to a readjustment in expectations about this ratio, which come from the new expectations about the future market value of the firm.

In cell "L4", the expression that appears is:

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=LN(kbar_0)-LN(kbar_1)
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that calculate the difference between the initial steady state and the new steady state when a disturbance occurs. That is, the deviation of the stock of capital from its new steady-state value.

If we place the cursor in cell "M3", the expression that appears is:

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=((R_0*Phi_0-(Alpha_0-1)*(R_0+Delta_0))/Phi_0)*K3
-((R_0+Delta_0)*(Alpha_0-1))*L3
```

which corresponds to the dynamic equation for the q ratio. Finally, if we place the cursor in cell "N3", the expression that appears is:

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=(1/Phi_0)*K3
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which corresponds to the dynamic equation for the capital stock.

3 Exercises

1. Suppose that an earthquake destroys 20% of the capital stock. Using the "Tobin.xlsx" spreadsheet, study the effects of this disturbance. (Hint: Change the initial value of the stock of capital in cell “J3” to 80% of the steady state value, "`Kbar_0*0.8`").

2. Study the effects of increase in the parameter α (for example from an initial value of 0.35 to a new value of 0.4 in cell "C12").