The dynamic IS-LM model

1 The model

We present here a dynamic version of the standard IS-LM model. As in the traditional static version of the model, it is considered the existence of two markets: the goods and service markets and the money market. The macroeconomic equilibrium in this setup is represented by the existence of equilibrium in both markets simultaneously. Additional to the IS equation and the LM equation presented in the static version, the model includes two additional equations accounting for the dynamic adjustment in output and prices.

The model economy is represented by the following four equations:

$$m_t - p_t = \psi y_t - \theta i_t \tag{1}$$

$$y_t^d = \beta_0 - \beta_1 (i_t - \Delta p_t^e) \tag{2}$$

$$\Delta p_t = \mu (y_t - y_t^n) \tag{3}$$

$$\Delta y_t = \upsilon (y_t^d - y_t) \tag{4}$$

where all variables are defined in logarithms[1], except the nominal interest

[1] Variables are defined in natural logarithms so that the equations are linear. For instance, the standard money demand equation is defined as,

$$\frac{M_t}{P_t} = Y_t^{\psi} \exp(-\theta i_t)$$

where M_t is the quantity of money, P_t is the price level, Y_t is the level of output, and i_t is the nominal interes rate. Basic rules for logarithms are $\ln(XZ) = \ln X + \ln Z$, $\ln(\frac{X}{Z}) = \ln X - \ln Z$, and $\ln X^k = k \ln X$. Using logarithms for the variables in levels (all variables except the nominal interes rate), we obtain the following linear money demand equation:

$$m_t - p_t = \psi y_t - \theta i_t$$

where lowercase letters are the logarithm of variables in levels.

rate, and where m denotes the quantity of money, p is the price, y is output, i is the nominal interest rate, y^d is the aggregate demand, and y^n is potential output. Δp^e represents expected inflation. Since we solve the model in a context of perfect foresight and under the assumption that expectations are rational, then we have that $\Delta p^e = \Delta p$. The symbol Δ defines the variation of the corresponding variable between two periods, where inflation is defined as

$$\Delta p_t = p_{t+1} - p_t \tag{5}$$

and output growth as[2]

$$\Delta y_t = y_{t+1} - y_t \tag{6}$$

Equation (1) is the equilibrium condition in the money market, where the real balances (left side), depend positively on the level of production (transaction money demand) and negatively on the nominal interest rate (speculative money demand). Equation (2) represents the aggregate demand of the economy, which depends positively on an autonomous component (which we assume reflects government spending), which we assume is an exogenous variable, and negatively of the real interest rate. The real interest rate is represented by the approximation to the Fisher equation and is obtained as the difference between the nominal interest rate and the expected inflation rate. The quantity of money is assumed to be exogenous.

Apart from the two equilibrium equations for the two markets, the model is also composed of two dynamic equations that indicate the dynamic behavior of two endogenous variables (price level and production level) over time. Equation (3) indicates how prices move over time based on the differences between the level of output and potential output, where potential output is exogenously given. This equation can be interpreted as a version of the

[2] We define

$$x_t = \ln X_t$$

where capital letters represent a variable in level and a lowercase letter represents the variable in logarithm. Hence, the growth rate for each variable, can be defined as:

$$\begin{aligned} \Delta x_t &= x_{t+1} - x_t = \ln X_{t+1} - X_t = \ln \frac{X_{t+1}}{X_t} \\ &= \ln\left(1 + \frac{X_{t+1} - X_t}{X_t}\right) \simeq \frac{X_{t+1} - X_t}{X_t} \end{aligned}$$

that is, the variation with respect to time in logarithmic terms is approximately equivalent to the growth rate of the variable in levels (for small enough growth rates).

Phillips curve. If the production level is greater than the potential, then this equation is positive, so that prices increase (positive inflation). On the contrary, if the production level is below the potential, the equation would have a negative sign, indicating that prices would decrease (deflation). Finally, equation (4) is similar but representing the dynamics of the production level. This expression indicates how changes in the level of output (the growth rate of the economy) moves depending on the differences between aggregate supply and demand. If the aggregate-demand level is higher than the production level, the expression would take a positive value, indicating that the level of production increases. On the contrary, if the production level is higher than the demand, then the expression would take a negative value, so that the production level of the economy would decrease.

All parameters (represented by greek letters) are defined in positive terms. The parameter ψ represents the elasticity of the real balances with respect to the level of production. θ is the semi-elasticity of the demand for money with respect to the nominal interest rate. It is a semi-elasticity because all the variables of the model are defined in logarithmic terms, except the interest rate, which, as it is a percentage, logarithms cannot be applied to it, since it is as if it were in that term. The parameter β_1 represents the elasticity of the level of aggregate demand with respect to the real interest rate, while β_0 is the autonomous component of aggregate demand, which we assume reflects public spending (public consumption).[3] The parameter μ is the speed of adjustment of prices to differences between the level of production and the level of potential production. Finally, the parameter υ indicates the speed of adjustment of the level of output in response to differences between the level of aggregate demand and the production level of the economy.

Solution To solve this model analytically, as a previous step to its numerical resolution, we will proceed by reducing the model to a system of two difference equations, in terms of the price and output. To obtain these two equations for inflation and output growth, we must first solve for the rest of the endogenous variables, that is, nominal interest rate and aggregate

[3]The constant of the aggregate demand function includes other component apart from government spending, such as autonomous consumption and autonous investment. To simplify, we assume that the constant of the aggregate demand function is a proxy for government expenditures.

demand. To obtain the nominal interest rate, we solve equation (1):

$$i = -\frac{1}{\theta}\left(m_t - p_t - \psi y_t\right) \tag{7}$$

to obtain the value of the nominal interest rate as a function of the endogenous variables to calculate (prices and output) and the exogenous variables (quantity of money). Next, we will solve for the aggregate demand. As we can see in equation (2), the nominal interest rate appears as a variable driving aggregate demand. Replacing equation (7) in (2), so that we obtain:

$$y_t^d = \beta_0 + \frac{\beta_1}{\theta}\left(m_t - p_t - \psi y_t\right) + \beta_1 \Delta p_t^e) \tag{8}$$

Assuming perfect foresight, that is, $\Delta p_t^e = \Delta p_t$, and substituting expression (3) into (8), we arrive to the equation that determines the aggregate demand of the economy:

$$y_t^d = \beta_0 + \frac{\beta_1}{\theta}\left(m_t - p_t\right) + \left(\beta_1\mu - \frac{\beta_1\psi}{\theta}\right) y_t - \beta_1 \mu y_t^n \tag{9}$$

To obtain the two dynamic equations that will determine the behavior of our economy, we have to replace these two endogenous variables (nominal interest rate and aggregate demand) in the adjustment equations of the endogenous reference variables (price and output). In the case of the dynamic equation for the price, these variables do not appear, so this first equation is exactly the same as that provided by the model (equation 3):

$$\Delta p_t = \mu(y_t - y_t^n) \tag{10}$$

Next, we obtain the dynamic equation for output. For this, we substitute the value obtained for the aggregate demand in the dynamic equation for output, such as:

$$\Delta y_t = \upsilon\left(\beta_0 + \frac{\beta_1}{\theta}\left(m_t - p_t\right) + \left(\beta_1\mu - \frac{\beta_1\psi}{\theta} - 1\right) y_t - \beta_1 \mu y_t^n\right) \tag{11}$$

Therefore, the model can be represented by the following system of two difference equations:

$$\begin{bmatrix} \Delta p_t \\ \Delta y_t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \mu \\ \frac{-\upsilon\beta_1}{\theta} & \upsilon(\beta_1\mu - \frac{\beta_1\psi}{\theta} - 1) \end{bmatrix}}_{A} \begin{bmatrix} p_t \\ y_t \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & -\mu \\ \upsilon & \frac{\upsilon\beta_1}{\theta} & -\upsilon\beta_1\mu \end{bmatrix}}_{B} \begin{bmatrix} \beta_0 \\ m_t \\ y_t^n \end{bmatrix} \tag{12}$$

where A is the matrix of parameters for the endogenous variables and B is the matrix of parameters of the exogenous variables.

Steady state Steady state of the economy is defined as the value for the endogenous variables for which they remain constant over time, that is, inflation and output growth are zero. Steady state is calculated by simply equating the dynamic equations of the system to zero, indicating that the variation over time of the endogenous variables is zero, so the variables are constant period to period. Using the model in matrix notation, the vector of the variables in steady state would be defined as:

$$\begin{bmatrix} \overline{p} \\ \overline{y} \end{bmatrix} = -A^{-1}B \begin{bmatrix} \beta_0 \\ m_t \\ y_t^n \end{bmatrix} \tag{13}$$

where steady state values are denoted by an upper bar on the variable. We start by inverting the matrix A. To do this, first calculate the adjugate of matrix A, $adj(A)$, being:

$$adj(A) = \begin{bmatrix} \upsilon(\beta_1\mu - \frac{\beta_1\psi}{\theta} - 1) & \frac{\upsilon\beta_1}{\theta} \\ -\mu & 0 \end{bmatrix} \tag{14}$$

and the traspose is,

$$adj(A)' = \begin{bmatrix} \upsilon(\beta_1\mu - \frac{\beta_1\psi}{\theta} - 1) & -\mu \\ \frac{\upsilon\beta_1}{\theta} & 0 \end{bmatrix} \tag{15}$$

where the determinant is:

$$\mid A \mid = \frac{\upsilon\beta_1\mu}{\theta}$$

so the negative of the inverse of matrix A is:

$$-A^{-1} = \begin{bmatrix} -\theta + \frac{\psi}{\mu} + \frac{\theta}{\beta_1\mu} & \frac{\theta}{\upsilon\beta_1} \\ -\frac{1}{\mu} & 0 \end{bmatrix}$$

Therefore, we obtain:

$$\begin{bmatrix} \overline{p}_t \\ \overline{y}_t \end{bmatrix} = -A^{-1}B\mathbf{z}_t = \begin{bmatrix} -\theta + \frac{\psi}{\mu} + \frac{\theta}{\beta_1\mu} & \frac{\theta}{\upsilon\beta_1} \\ -\frac{1}{\mu} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\mu \\ \upsilon & \frac{\upsilon\beta_1}{\theta} & -\upsilon\beta_1\mu \end{bmatrix} \begin{bmatrix} \beta_0 \\ m_t \\ y_t^n \end{bmatrix}$$

and by multiplying the matrices $-A^{-1}B$ we get:

$$\begin{bmatrix} \overline{p}_t \\ \overline{y}_t \end{bmatrix} = \begin{bmatrix} \frac{\theta}{\beta_1} & 1 & -\psi - \frac{\theta}{\beta_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ m_t \\ y_t^n \end{bmatrix}$$

Operating, we obtain the following expressions that define the steady state of the economy:

$$\overline{p} = \frac{\theta\beta_0}{\beta_1} + m - (\psi + \frac{\theta}{\beta_1})y^n \tag{16}$$

$$\overline{y} = y^n \tag{17}$$

Eigenvalues The stability of the system is determined by calculating $|A - \lambda I| = 0$, where A is the matrix of parameters defined above, λ represents the vector of eigenvalues and I is the identity matrix, being:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{18}$$

In our case, we would have:

$$\begin{aligned} Det \begin{bmatrix} 0 & \mu \\ \frac{-\upsilon\beta_1}{\theta} & \upsilon(\beta_1\mu - \frac{\beta_1\psi}{\theta} - 1) \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \\ Det \begin{bmatrix} -\lambda & \mu \\ \frac{-\upsilon\beta_1}{\theta} & \upsilon(\beta_1\mu - \frac{\beta_1\psi}{\theta} - 1) - \lambda \end{bmatrix} &= 0 \end{aligned} \tag{19}$$

Eigenvalues are,

$$\lambda_1, \lambda_2 = \frac{\upsilon(\beta_1\mu - \frac{\beta_1\psi}{\theta} - 1) \pm \sqrt{\left[\upsilon(\beta_1\mu - \frac{\beta_1\psi}{\theta} - 1)\right]^2 - \frac{4\upsilon\beta_1\mu}{\theta}}}{2} \tag{20}$$

For stability, the term $\upsilon(\beta_1\mu - \frac{\beta_1\psi}{\theta} - 1)$ must be positive. In this case, the two eigenvalues are negative, and the modulus of eigenvalues plus one is less than unity. Hence, the solution of the system shows global stability.

2 Taking the model to Excel

The Excel file for this model is named "ISLM.xlsx", The Model tab. Columns "F", "G", "H" and "I" show the value of each of the endogenous variables (prices, output, aggregate demand and nominal interest rate) at each moment of time. The calibrated values of the parameters appear in cells "B10" to "B14". The initial values of the exogenous variables appear in cells "B17", "B18" and "B19", which we have named "`m_0`", "`Beta0_0`" and "`ypot_0`", respectively. The initial steady-state values appear in cells "B22" and "B23". Cells "C22" and "C23" show the new steady state in the case where a disturbance occurs (change in the exogenous variables). The value of the eigenvalues is given in rows 26 and 27. In cells "B26" and "B27", the real part is shown, while the imaginary part is shown in cells "C26" and "C27". Finally, the root module is shown in cells "B30" and "B31".

If we place the cursor in cell "F3" this expression appears:

```
=(Theta*Beta0_0)/Beta1+m_0-(Psi+Theta/Beta1)*ypot_0
```

which is simply the expression corresponding to the initial steady-state value of the price level. Alternatively, we could simply enter the reference to cell "B22", in which we have calculated the corresponding steady-state value. The remaining rows in this column simply contain the value of the price level in the previous moment plus the change produced in that price level. Thus, cell "F4" contains the expression "`=F3+J3`", where "F3" refers to the price level of the previous period and "J3" to the change in the price level. This expression is copied into the remaining rows of that column.

On the other hand, if we place the cursor in cell "G3" it contains the expression: "`=ypot_0`", that is, the initial steady-state value of the production level that corresponds to the level of potential production. Alternatively, we can simply enter cell "B23". In cell "G4", the expression "`=G3+K3`" appears in which we define the production level of each period as the previous one plus the change experienced in it. Column "H" contains the aggregate demand values. If we place ourselves in cell "H3", we see that this expression appears:

```
=Beta0_0-Beta1*(I3-J3)
```

which corresponds to the aggregate demand equation of the model, in which aggregate demand depends negatively on the real interest rate, which we have defined as the difference between the nominal interest rate and inflation. This same expression appears in the following cells in this column. Column "I" contains the nominal interest rate values. Thus, cell "I3" contains the following expression:

```
=-1/Theta*(m_0-F3-Psi*G3)
```

which is the equation resulting from solving for the interest rate from the money demand equation. If we place ourselves in cell "I4", the expression that appears is:

```
=-1/Theta*(m_1-F4-Psi*G4)
```

which refers to the new amount of money at time 1. This expression is the same as that which appears in the following rows of this column.

Columns "J" and "K" show the variations in prices and output, that is, they define the value of inflation and the growth of production in each period. In this case, we must introduce the corresponding equations that determine the behavior of both variables. If we place ourselves in cell "J3" we see that it contains the expression:

```
=Mi*(G3-ypot_0)
```

while cell "J4" contains the expression:

```
=Mi*(G4-ypot_1)
```

being this same expression the one that appears in the following cells, since it is possible that we want to analyze the effects of an alteration in the potential production of the economy. On the other hand, if we place ourselves in cell "K3", we see that it contains the expression:

```
=Ni*(H3-G3)
```

which corresponds to the dynamic equation for output. As we can see in the spreadsheet, we can enter the initial expression given by the model, since we will also calculate the corresponding value of the aggregate demand at each moment of time. If all calculations are correct, columns "J" and "K", where the change of each variable appears, must be zeros.

Columns "L" to "O" show the variables in levels. This is done by applying the antilogarithm to the variables in logs, except to the nominal interest rate. For example, in cell "L3" we find the expression `"=exp(F3)"`, which transforms the log of the price into the price level. Finally, columns "P" to "S" show the deviation of the variables in levels, in percentage points, with respect to the initial steady state (basic points for the nominal interest rate)

3 Exercises

1. Using the "ISLM.xlsx" spreadsheet, The Model tab, study the effects of an increase in public spending, represented by β_0. Specifically, suppose that (the log of) public spending increases by 0.1 units. To do this, change the value of cell "C18" from 25 to 25.1 (this is equivalent to an increase of approximately 10.5% in the level of public spending). What does the dynamic IS-LM model tell us about the effects of an expansionary fiscal policy?

2. Study what the effects of an increase in the level of potential production are. Suppose, for example, that (the log of) potential increases to 20.2 (this is equivalent to an increase of approximately 22% in the level of potential output). Use the "ISLM.xlsx" spreadsheet, The Model tab, to determine the effects of this change. To do this exercise, change the corresponding value in cell "C19".

3. Repeat the exercise carried out in the spreadsheet (increase in (the log of) money supply from 5 to 5.1), but assuming that prices are more flexible. To do this, you have to increase the value of the adjustment parameter in the price equation, μ (for example, from the initial value of 0.01 to a value of 0.02).